Calculation of Eigenvalue and Eigenvector Derivatives for Nonlinear Beam Vibrations

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A beam with immovable ends under large amplitude oscillations is axially subjected to an amplitude-dependent stretching force which is induced by the geometrical nonlinearity. The design sensitivity analysis for such a beam is investigated in this study. Two variational formulations pertaining to nonlinear beam vibration based on two different approximations of nonlinear stretching force are presented. The analytical equations of design sensitivity analysis for these nonlinear equations are then derived. Finally, a computational scheme using the finite element method is developed to calculate the design sensitivity derivatives of the eigenvalue and eigenvector numerically. The numerical results show that the proposed algorithm for calculating design derivatives of nonlinear eigenvalues and eigenvectors is valid, and, compared with the direct finite difference method, the proprosed algorithm is efficient.

I. Introduction

THE major thrust of this paper is to derive an analytical formulation and develop a numerical scheme to compute the design derivatives of the eigenvalue and eigenvector for the vibration of beams with large amplitude.

With increasing emphasis on structural efficiency, many minimum-weight or light-weight designed structural systems are subjected to large deflection which may cause operational difficulty. In many cases, it is necessary for the structural systems to be analyzed, using the nonlinear theory, in order to realize the effects of large deformation. For example, experimenal results[1-3] have shown that high sound pressure levels (SPL) produce nonlinear behavior in various aircraft panels. The linear structural analysis predicts the root-mean-squared (RMS) strain/stresses and RMS deflection well above those of the experimental results, and the frequencies of vibration well below those of the experimental results. It is then apparent that the linear analysis lacks the capability to predict accurately panel service life. Thus, in order to have a reliable product, a design engineer should be aware of the relationship between the nonlinear behavior and design parameters in the design process. Nevertheless, the analysis of a structural component with large deformation is an iterative and time-consuming process. Consequently, the requirement of many reanalyses to modify the structural system may make an iterative design process prohibitive. In this case, the formulation of design sensitivity analysis becomes a necessary means, because of its efficiency for the prediction or approximation of the change of system responses when the design undergoes small modification.

Many methods are available to calculate design derivatives of linear eigenproblems with arbitrary matrices. On the other hand, although nonlinear vibration has received considerable attention in the past decades, only a limited number of publications are available regarding the design derivatives of the eigenvalues and eigenvectors for a nonlinear vibration problem.

This paper presents an initial attempt to formulate systematically the equation for the design sensitivity analysis of a nonlinear vibration problem. The structural system to be considered is a beam with immovable ends under nonlinear vibration. The geometrical nonlinearity results from the interaction between the axial deformation and lateral deflection. In contrast to the linear eigenproblem, the eigenvalue and eigenvector of such a beam are amplitude dependent. Moreover, the design derivative of an eigenvalue is a function of the eigenvector design derivative.

The equation of motion regarding the nonlinear vibration of beams with immovable ends can be described by the weakly nonlinear theory. In this theory, the deformation is characterized by large displacement and small strain. The equation of motion is a nonlinear partial differential equation in both time and space. In general, the technique of variable separation does not work for this equation. To solve this equation, the approaches commonly used can be broadly classified as being either of the assumed space mode (ASM) type or the assumed time mode (ATM) type. The aim of these approaches is to reduce this nonlinear partial differential equation to a nonlinear ordinary differential equation being either in time or in space.

The nonlinear free vibration of a beam simply supported on immovable hinges was considered first by Woinowski-Krieger.[4] He obtained an ordinary differential equation in time by substituting an assumed sinusoidal space mode in the equation of motion. The resulting differential equation was then solved analytically to obtain the period of oscillation or frequency. For other support conditions, such as clamped-clamped and simple supported-clamped boundary conditions, some approximate mode shapes are used to convert the partial differential equation of motion into an ordinary differential equation of motion in time.[5,6]

The research done by Ray and Bert,[7] as well as Srinivasan,[8] can be categorized as the ATM approach. In their approaches, the time part of the problem is assumed to be a sine harmonic function of frequency and is approximated by ignoring high-order harmonic terms. The resulting differential equation in the space variable is solved analytically.

The ATM and research of Ray and Bert[7] and Srinivasan[8] open the door for a more general computational scheme: the Finite Element Method. Mei[9,10] was the first to solve the problems of large amplitude vibrations by using a finite element displacement method. His work was based on the assumption

Presented as Paper 86-0963 at the AIAA/ASME/ASCE/AHS 27th Structures, Structural Dynamics, and Materials Conference, San Antonio, Texas, May 19–21, 1986; received Oct. 1, 1986; revision received Aug. 7, 1987.

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that the axial tension remains constant within a finite element, and the displacement is a periodic function of time with circular frequency. Later developments of finite element approaches center on how to define the time function and how to deal with the nonlinear term associated with axial tension.[11-14]

In early research, the stretching force in the formulation of nonlinear vibration was only expressed in terms of lateral deflection, because the inplane inertia was not taken into account. The nonlinear stretching force was originally assumed to be constant within a element.[9,10] Instead of dealing with stretching force, an alternative approach was proposed[11,12] based on an appropriate linearization of the strain-displacement relations. Using the last approach, the finite element method has been developed[15] to solve the problems of nonlinear vibration in which the inplane inertia is included.

Design sensitivity analysis plays a central role in structural optimization, since virtually all gradient-based optimization methods are required to compute the design derivatives of structural response quantities, with respect to design variables. In addition, the design derivative is very useful in design, because it provides a quick estimate of the change of the responses due to the given change of the current design variables. Most publications for the design sensitivity analysis of eigenproblems are associated with linear vibration. The algorithms for calculating the design gradient of linear eigenproblems have already been implemented in commercial computer codes such as EAL and MSC/NASTRAN. Furthermore, it is becoming a common practice for engineers to apply the information of design gradients of eigenvalues and eigenvectors to the design problems where the dynamic response and dynamic stability are considered, such as in machine design, or in the controller design for the control of structure. The book of Haug, Choi and Komkov[16] provides the mathematical theory of the design derivatives of linear eigenvalues problems which can be formulated as distributed parameter problems. On the other hand, computational algorithms have been developed for the calculation of design gradients based on the discrete forms of linear eigenvalue problems.[17-19] Recently, Baldwin and Hutton[20] reviewed different methods for structural dynamics modifications. The linear eigenvalue design sensitivity analysis is considered as one of the techniques based on small modification and is best suited to finite modifications of general structural parameters for changes on the order of 5%.

As for the design sensitivity analysis of nonlinear eigenproblems, only limited literature is available. Rao et al.[21,22] applied an optimality criteria method to obtain the optimum configuration of beam and plate structures executing large amplitude oscillations, subjected to a frequency constraint. They presented a simple relation between the nonlinear geometrical matrix and the design variables. The design variables are the thickness of the beam and the thickness of the plate. Based on this given relation, the eigenvalue derivative can be obtained in the same fashion as the one obtained for the linear vibration problem. An interesting feature for the nonlinear vibration problem is that the eigenvalue derivative is related to the eigenvector derivative. This, however, has not been indicated in the work of Rao and his colleagues.

II. Formulation of Nonlinear Vibration of Beam with Immovable Ends

The formulation of nonlinear vibration of a beam with immovable ends is presented here. With appropriate assumptions, it can be shown that the final form of the governing differential equation for the nonlinear vibbration of a beam with immovable ends can be described in terms of the transverse displacement alone. This is done by suitable manipulations of the differential equations and boundary conditions which arise from a variation of the Lagrangian of the system.

The geometry of a beam structure with a rectangular cross-section is shown in Fig. 1. It is assumed that the dimensions of

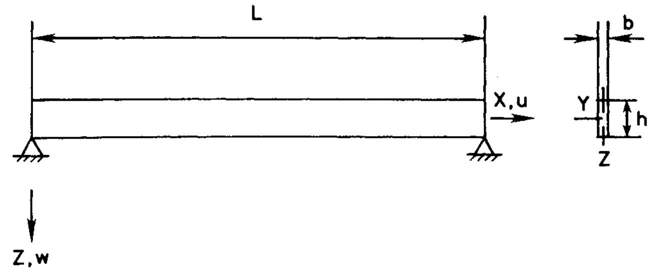

Fig. 1 The geometry of a beam structure.

depth h and width b are both much smaller than the length L. Two displacement quantities, the axial displacement u and the transverse displacement w are used to define the deformation of a beam along the neutral axis. For a beam, the general expression for the nonlinear relation of the strain-displacement can be found as:

$$\epsilon_{xx} = u,_x - \rho w,_{xx} + \tfrac{1}{2} w,_x^2$$

$$\epsilon_{xy} = \epsilon_{yy} = 0$$

where ρ is the distance measured from the neutral axis. Using Hooke's law and the above equation, one can express the strain energy in terms of displacements as

$$U = \tfrac{1}{2}\int_0^L EIw,_{xx}^2 \mathrm{d}x + \tfrac{1}{2}\int_0^L EA(u,_x + \tfrac{1}{2}w,_x^2)^2 \mathrm{d}x \tag{1}$$

where the notation I is the moment of inertia with respect to the neutral axis of the beam's cross section, the notation E denotes the modulus of elasticity, and the notation A denotes the cross-sectional area of the beam.

The kinetic energy T is written as

$$T = \tfrac{1}{2}\int_0^L m(u,_t^2 + w,_t^2)\mathrm{d}x \tag{2}$$

where the notation m denotes the mass per unit length and the notation $(\cdot),_t$ represents the partial derivative of the quantity $(\cdot)$ with respect to t. It is assumed that the kinetic energy due to inplane displacement is small compared to that due to transverse displacement during flexural vibration. The approximation is valid if the ratio of g/L is small where g is the radius of gyration of the cross sectional area.[23] Consequently, the inplane inertia can be neglected. Therefore, the kinetic energy T is simplified as

$$T = \tfrac{1}{2}\int_0^L mw,_t^2 \mathrm{d}x$$

Based on Hamilton's principle, the variation of the Lagrangian $L = U - T$ provides the governing equations of motion and boundary conditions of a beam under large amplitude vibration:

$$(EIw,_{xx}),_{xx} - [EA(u,_x + \tfrac{1}{2}w,_x^2)w,_x],_x + mw,_{tt} = 0 \tag{3}$$

$$[EA(u,_x + \tfrac{1}{2}w,_x^2)],_x = 0 \tag{4}$$

with the boundary condition at $x=0$ and L. It is observed from the integration of Eq. (4) that

$$EA(u,_x + \tfrac{1}{2}w,_x^2) = N$$

where N is an integration constant. Therefore, one has

$$u,_x = \frac{N}{EA} - \tfrac{1}{2}w,_x^2 \tag{5}$$

The integration of Eq. (5) over the length of a beam leads to

$$u(L)-u(0)=\frac{N}{E}\int_0^L \frac{1}{A}\mathrm{d}x-\int_0^L \tfrac{1}{2} w,_x^2\mathrm{d}x \tag{6}$$

It is at this moment that the important boundary conditions associated with a beam with immovable ends is introduced. For example, the kinematic boundary conditions of a simply supported beam with immovable ends are simply given as

$$u(0)=u(L)=0 \tag{7}$$

and

$$w(0)=w(L)=0 \tag{8}$$

On the other hand, the kinematic boundary conditions of a clamped-clamped beam with immovable ends are

$$u(0)=u(L)=0 \tag{9}$$

$$w(0)=w(L)=0 \tag{10}$$

$$w,_x(0)=w,_x(L)=0 \tag{11}$$

Because of the boundary conditions of Eqs. (7-11) (for beams with immovable ends), Eq. (6) can be reduced to,

$$N=\frac{E}{2Q}\int_0^L w,_x^2\mathrm{d}x=EA(u,_x+\tfrac{1}{2}w,_x^2) \tag{12}$$

where the notation Q is the reciprocal of the volume of the beam, i.e.,

$$Q=\int^L \frac{1}{A}\,\mathrm{d}x \tag{13}$$

The combination of Eqs. (3), (4), and (12) allows the large amplitude vibration of an immovably supported beam to be described by the following differential equation in the transverse displacement alone:

$$(EIw,_{xx}),_{xx}-Nw,_{xx}+mw,_{tt}=0 \tag{14}$$

with boundary conditions given in Eqs. (7-11) and where the N is defined as

$$N=\frac{E}{2Q}\int_0^L w,_x^2\mathrm{d}x \tag{15}$$

The above expression indicates that N represents an averaged nonlinear stretching force which is a functional of transverse displacement alone.

Due to the geometrical nonlinearity, there is no one preferred way to convert the equation of motion, Eq. (14), into a nonlinear eigenproblem. In the following sections, however, two different nonlinear eigenproblems are derived from Eqs. (1,2) [or Eq. (14)] based on two different assumptions proposed in Refs. (7) and (9), respectively.

Ray and Bert[7] assumed that the time mode be of the simple form:

$$T(t)=\sin\omega t$$

Thus, Eq. (14) becomes an equation with $\sin\omega t$ and $\sin^3\omega t$ terms. The $\sin^3\omega t$ term can be expanded by using the trigonometric identity:

$$\sin^3\omega t=(3/4)\sin\omega t-(1/4)\sin 3\omega t$$

Since only the fundamental frequency is of interest, the higher-harmonic term, $\sin 3\omega t$, can be dropped. The equation of motion, Eq. (14), can then be converted into a problem of nonlinear vibration:

$$(EIw,_{xx})-\tfrac{3}{4}Nw,_{xx}-\lambda mw=0 \tag{16}$$

where λ is the square of angular frequency ω and the N is the stretching force. Note that the w in Eq. (16) is not the transverse displacment given in Eq. (14). Instead, it is defined as the eigenmode of the nonlinear vibration, Eq. (16). Moveover, the boundary conditions for Eq. (16) are the same as those of Eq. (14).

To obtain the variational form for nonlinear vibration, one may multiply Eq. (6) by a trial function $z(x)$ and integrate the resulting product over the length of the beam. Let the kinematically admissible function space Z be defined as a set composed of functions which satisfy the boundary conditions Eqs. (7-11), and which have enough differential regularity. Consequently, the nonlinear engenmode $w\epsilon Z$ is the solution of the variational equation:

$$\int_0^L (EIw,_{xx}z,_{xx}+\tfrac{3}{4}Nw,_x z,_x-\lambda mwz)\,\mathrm{d}x=0 \tag{17}$$

for any trial function z which belongs to Z.

Finally, using the following definitions of bilinear forms,

$$a(w,z)\equiv\int_0^L EIw,_{xx}z,_{xx}\,\mathrm{d}x,$$

$$d(w,z)\equiv\int_0^L w,_x z,_x\,\mathrm{d}x,$$

$$c(w,z)\equiv\int_0^L mwz\,\mathrm{d}x \tag{18}$$

one can abbreviate Eq. (17) to:

$$a(w,z)+\frac{3E}{8Q}\,\mathrm{d}(w,w)\cdot \mathrm{d}(w,z)-\lambda\cdot c(w,z)=0 \tag{19}$$

Another assumption regarding the formulation of nonlinear beam vibration was introduced by Mei[9] who averaged the nonlinear stretching force over the length of a beam element. According to Mei's assumption, the variational form of nonlinear vibration can be written as

$$\sum_{i=1}^{\mathrm{NE}}\left[\int_0^{\ell_i}(EI_iw,_{xx}z,_{xx}+N_iw,_x z,_x-\lambda m_iwz)\,\mathrm{d}x\right]=0 \tag{20}$$

where the notation NE is the total number of finite elements, the notation ℓ_i denotes the length of the ith beam element, and the average axial tension N_i is defined as follows:

$$N_i\equiv\tfrac{1}{2}\int_0^{\ell_i}\frac{EA_i}{2}w,_x w,_x\,\mathrm{d}x \tag{21}$$

It is noted that the nonlinear stretching force defined above is only calculated over a length of a beam element, while the stretching force given in Eq. (15) is calculated over the entire length of a beam. Moreover, the leading coefficient, 1/2, in Eq. (21) refers to the average effect of the harmonic motion on the nonlinear stretching force. The detailed discussion of the differences between the nonlinear eigenproblems, Eq. (19) and Eq. (20), can be found in Refs. (7) an (24).

Numerical examples are presented in Sec. V to validate the formulations discussed above.

III. Design Sensitivity Analysis for Nonlinear Eigenproblems

A variational approach has been developed in the literature, for example Ref. (16), to calculate the design derivatives of linear eigenproblems. With little modification, the developed variational approach can be extended here to obtain the design derivatives of nonlinear eigenproblems. Nevertheless, there are significant differences between the design derivatives of linear and nonlinear eigenproblems. For example, the equations for computing the design derivatives of the nonlinear eigenvalue and the eigenvector are interrelated for a nonlinear eigenproblem but are not related for a linear eigenproblem.

This section is devoted to the mathematical formulation of the design sensitivity analysis for the nonlinear vibration problem formulated by Eq. (19). The numerical implementation of the derived formulation is to be discussed in the next section. Since they are very similar to those presented here, the mathematical derivation and numerical results of the design sensitivity analysis based on the nonlinear eigenproblem formulated by Eq. (20) will be outlined in the Appendix.

Let $b(x)$ be defined as the design parameter that relates to the cross-sectional area. The design derivative of the expression of Eq. (19) yields the following equation:

$$\left[\frac{3E}{8Q}\,\mathrm{d}(w,w)\cdot\mathrm{d}(w_b,z)+a(w_b,z)-\lambda\cdot c(w_b,z)\right]+\frac{3E}{4Q}\,\mathrm{d}(w,w_b)\cdot\mathrm{d}(w,z)=\lambda_b\cdot c(w,z)+\lambda\cdot c_b(w,z)+\frac{3EQ_b}{8Q^2}\,\mathrm{d}(w,w)\cdot\mathrm{d}(w,z)-a_b(w,z) \qquad (22)$$

where Q_b is the design derivative of Q defined in Eq. (13):

$$Q_b=-\int_0^L\frac{A_b}{A^2}\,\mathrm{d}x$$

Note that the terms in the first bracket of the left side of Eq. (22) are zero for $z=w$ based on Eq. (19) and the fact that $w_b \epsilon Z$. Hence, Eq. (22) can be simplified as,

$$\lambda_b=\frac{1}{c(w,w)}\left[\frac{3E}{4Q}\,\mathrm{d}(w,w)\cdot\mathrm{d}(w,w_b)-\frac{3EQ_b}{8Q^2}\,\mathrm{d}(w,w)\cdot\mathrm{d}(w,w)+a_b(w,w)\lambda\cdot c_b(w,w)\right] \qquad (23)$$

by replacing z by w. The foregoing equality is a linear equation consisting of both w_b and λ_b. It is evident that Eq. (23) indicates an interdependence between the eigenvalue derivative λ_b and the eigenfunction derivative w_b. In other words, λ_b cannot be solved without knowing the information of w_b. Such an interdependence has not been found in linear eigenvalues problems.

To calculate w_b, one may replace λ_b in Eq. (22) by Eq. (23) to obtain:

$$\frac{3E}{4Q}\,\mathrm{d}(w,w_b)\cdot\left[\mathrm{d}(w,z)-\frac{c(w,z)}{c(w,w)}\cdot\mathrm{d}(w,w)\right]+\frac{3E}{8Q}\,\mathrm{d}(w,w)\cdot\mathrm{d}(w_b,z)+a(w_b,z)-\lambda\cdot c(w_b,z)=\frac{3EQ_b}{8Q^2}\,\mathrm{d}(w,w)\cdot\left[\mathrm{d}(w,z)-\frac{c(w,z)}{c(w,w)}\cdot\mathrm{d}(w,w)\right]+\lambda\cdot c_b(w,z)-\lambda\frac{c(w,z)}{c(w,w)}\cdot c_b(w,w)-a_b(w,z)+\frac{c(w,z)}{c(w,w)}\cdot a_b(w,w) \qquad (24)$$

The above equation is a linear equation of w_b. It is evident that both sides of Eq. (24) are identically zero for any arbitrary w_b, if z is replaced by w. The existence of infinite nonzero homogeneous solutions of Eq. (24) with $z=w$ indicates that the bilinear form on the left side of Eq. (24) is not a positive definite operator over the function space Z. Because Eq. (24) is trivially satisfied for $z=w$, one may restrict the kinematically admissible function space Z to a subspace W that is c-orthogonal to w. In other words, the kinematically admissible function space Z can be spanned by the one dimensional subspace $\{w\}$ and the subspace W, i.e., $Z=\{w\}\oplus W$, where $\{w\}$ is the one-dimensional subspace of Z spanned by w and $W=\{v\epsilon Z: c(v,w)=0\}$. The notation $\oplus$ means that every function in Z can be uniquely written in the form

$$z=\alpha w+v,\ \text{for}\ \alpha\epsilon R^1\ \text{and}\ v\epsilon W \qquad (25)$$

Moreover, because the design derivative of eigenfunction w_b satisfies the kinematical boundary conditions, Eqs. (7-11), w_b belongs to Z. Thus, according to Eq. (25), w_b can be uniquely expressed as,

$$w_b=\alpha w+\bar{w},\ \text{for}\ \alpha\epsilon R^1\ \text{and}\ \bar{w}\epsilon W \qquad (26)$$

Since Eq. (24) is valid for all $z\epsilon Z$, every element of Z can be also uniquely written in the form of Eq. (25). Furthermore, because Eq. (24) is trivially satisfied for $z=w$, Eq. (24) can be reduced to find *a* $\bar{w}\epsilon W$, such that

$$\frac{3E}{4Q}\,\mathrm{d}(w,\bar{w})\cdot\mathrm{d}(w,v)+\frac{3E}{8Q}\,\mathrm{d}(w,w)\cdot\mathrm{d}(\bar{w},v)+a(\bar{w},v)-\lambda\cdot c(\bar{w},v)=\frac{3EQ_b}{8Q^2}\,\mathrm{d}(w,w)\cdot\mathrm{d}(w,v)+\lambda\cdot c_b(w,v)-a_b(w,v),\quad \text{for all}\quad v\epsilon W \qquad (27)$$

To obtain a unique solution $\bar{w}$ for the above equation, one has to show that the bilinear form on the left-hand side of Eq. (27) is positive definite on W.

One may treat the axial force $3E/8Q[\mathrm{d}(w,w)]$ as a fixed scalar ϕ which is evaluated at the exact solution of eigenfunction w. As a result, the nonlinear eigenvalue problem, Eq. (19), is equivalent to the minimization of the Rayleigh's Quotient over $u\epsilon Z$:

$$\lambda=\underset{u\epsilon Z}{\mathrm{Min}}\ \frac{\phi\cdot\mathrm{d}(u,u)+a(u,u)}{c(u,u)} \qquad (28)$$

The unique solution of the above minimization problem is the eigenfunction w which belongs to Z. The reason is that the Rayleigh Quotient has a stationary point at $w\epsilon Z$, and that the Rayleigh Quotient is a convex functional defined in the kinematically admissible function space Z.

Hence, for any nonzero $v\epsilon W c Z$, one has the following inequality:

$$\phi\cdot\mathrm{d}(v,v)+a(v,v)-\lambda\cdot c(v,v)>0 \qquad (29)$$

The inequality sign is a result of Eq. (28) and the definition that $v\notin\{w\}cZ$. Furthermore, because of the positiveness of the term:

$$\frac{3E}{4Q}\,\mathrm{d}(w,v)\cdot\mathrm{d}(w,v)>0 \qquad (30)$$

the summation of Eqs. (29) and (30) is always positive on the subspace space W, i.e.,

$$\frac{3E}{4Q}\,d(w,v)\cdot d(w,v)+\phi\cdot d(v,v)+a(v,v)-\lambda\cdot c(v,v)>0 \tag{31}$$

Equation (31) shows that indeed the bilinear form on the left side of Eq. (27) is positive definite on W. Thus, Eq. (27) has a unique solution for $\bar{w}$ of the design sensitivity of the eigenfunction w corresponding to the smallest nonlinear eigenvalue λ.

Once $\bar{w}$ is determined, one can obtain the eigenfunction design derivative w_b from the relation, Eq. (26),

$$w_b=\alpha w+\bar{w}$$

and calculate the corresponding design derivative λ_b from Eq. (23), subsequently. It now remains to determine the coefficient α in Eq. (26).

The coefficient α is usually found by considering the normalization of the corresponding eigenfunction in the linear case. The same procedure may be pursued for the nonlinear case. However, the eigenfunction is usually normalized according to the ℓ^∞ norm for the nonlinear case, i.e., $\gamma=\max|w(x)|$; for all $0\le x\le L$ where the constant γ represents the assigned amplitude. This practice causes difficulty for the computation of the design derivative because the location of the point x_0 where $|w(x)|$ is maximized may also change due to the change of the design variables. In other words, the position x_0 depends on b as well, i.e., $x_0=x_0(b)$. For simplicity, $w(x)$ is assumed to have the maximum value at a single point x_0 in this study. Thus, the following equations are always true for the different designs.

$$w(x_0)=\gamma, \qquad \left.\frac{dw(x)}{dx}\right|_{x_0}=w'(x_0)=0 \tag{32}$$

Taking the design derivatives of the above equalities over design variable b, one has

$$\begin{aligned}0&=\frac{dw[b,x_0(b)]}{db}\\&=w_b(x_0)+w'(x_0)x_{0b}\\&=w_b(x_0)\end{aligned} \tag{33}$$

$$\begin{aligned}0&=\frac{dw'[b,x_0(b)]}{db}\\&=w_b'(x_0)+w''(x_0)x_{0b}\end{aligned} \tag{34}$$

where the notation x_{0b} is defined as the design derivative of position x_0, i.e., $x_{0b}=dx_0/db$. Substituting Eq. (26) for w_b and w_b' in the above equations, the coefficient α and the design derivative x_{0b} can then be calculated as follows:

$$\alpha=-\frac{\bar{w}(x)}{w(x_0)} \tag{35}$$

$$x_{0b}=\frac{dx_0}{db}=-\frac{\bar{w}'(x_0)}{w''(x_0)} \tag{36}$$

In short, it can be concluded that the design derivative of eigenfunction w_b can be calculated completely as

$$w_b=\alpha w+\bar{w}$$

where α is given in Eq. (35), w is the solution for a nonlinear eigenvalue problem, Eq. (19), and $\bar{w}$ is the solution for a linear equation, Eq. (27). Moreover, the design derivative eigenvalue λ_b can be determined based on Eq. (23).

IV. Finite Element Method

To start with the finite element method, a general discretized element of a beam structure is shown in Fig. 2. The notations D_i and S_i denote the transverse displacement and the slope, respectively, at the node i. Each node of the beam element has two degrees of freedom. The interpolation functions of the beam element are defined by cubic Hermite polynomials for $0\le x\le \ell_i$.

Using the standard assembly prodedure, the global stiffness matrix $[K]$, the mass matrix $[M]$, and the geometrical matrix $[G]$ can be assembled and modified according to Eq. (18) and the kinematical boundary conditions, to represent the characteristics of the entire beam.

Using the global matrices $[K]$, $[M]$ and $[G]$, the variational equation of Eq. (19) can be written in a matrix expression as

$$\left[\frac{3E}{8Q}(\{w\}^T[G]\{w\})\cdot[G]+[K]\right]\{w\} - \lambda[M]\{w\}=0 \tag{37}$$

It is clearly indicated that the above equation is nonlinear in terms of eigenvector $\{w\}$. An iterative scheme[9] could be introduced here to linearize the matrix equation. Let the superscript denote the iteration number. At the nth iteration, Eq. (37) is approximated by

$$\left[\frac{3E}{8Q}(\{w^{n-1}\}^T[G]\{w^{n-1}\})\cdot[G]+[K]\right]\{w^n\} - \lambda^n[M]\{w^n\}=0 \tag{38}$$

In other words, the nonlinear stretching term is approximated by using the eigenvector defined in the previous iteration. When $n=1$, the vector $\{w^\circ\}$ is taken as a null vector. Equation (38) is a linear eigenproblem for the eigenvalue λ^n and the eigenvector $\{w^n\}$ which can be easily solved by using the subspace iteration method.[25] However, the eigenvector $\{w^n\}$ is not normalized with respect to the mass matrix $[M]$ in our study. Instead, the eigenvector $\{w^n\}$ is normalized in such a

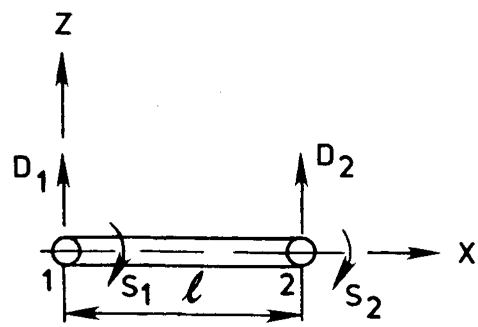

Fig. 2a A beam element.

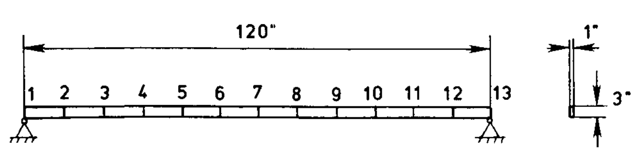

Fig. 2b The finite element model of a nonlinearly vibrating beam.

way that the maximum amplitude of the eigenvector is scaled to a given magnitude γ. This is one of the differences between the computational procedures for linear and nonlinear vibrations. This process is repeated until convergence is achieved for the nonlinear eigenvalue and eigenvectors, to the required accuracy. The convergence criteria is defined as the square root of the difference of eigenvectors, i.e.,

$$\sqrt{(\{w^n\}-\{w^{n-1}\})^T\cdot(\{w^n\}-\{w^{n-1}\})}<\epsilon$$

where the notation ϵ is a small number. In this study, the ϵ is taken as 10^{-16}.

The first step of the design sensitivity analysis is to solve $\{\bar{w}\}$ according to the variational formula of the design sensitivity analysis, Eq. (27). This equation can be written in a vector form for solving $\{\bar{w}\}$ as:

$$\left\{\frac{3E}{4Q}[G]\{w\}\cdot\{w\}^T[G]+\frac{3E}{8Q}\{w\}\cdot[G]+[K]-\lambda[M]\right\}\{\bar{w}\}=\frac{3\,EQ_b}{8Q^2}\{w\}^T[G]\{w\}\cdot[G]\{w\}+\lambda m,_b[M]\{w\}-EI,_b[K]\{w\} \tag{39}$$

Note that $[G]$, $[M]$ and $[K]$ are symmetric, and the leading coefficient of $\{\bar{w}\}$ on the left-hand side of the above equation is also symmetric. To abbreviate the notations, the following definitions are introduced:

$$[A]\equiv\frac{3E}{4Q}[G]\{w\}\cdot\{w\}^T[G]+\frac{3E}{8Q}\{w\}^T[G]\{w\}\cdot[G]+[K]-\lambda[M]$$

$$\{F\}\equiv\frac{3EQ_b}{2Q^2}\{w\}^T[G]\{w\}\cdot[G]\{w\}+\lambda m,_b[M]\{w\}-EI,_b[K]\{w\}$$

Therefore, Eq. (39) can be expressed simply as

$$[A]\{\bar{w}\}=\{F\} \tag{40}$$

It has been shown that the above equation provides a unique solution $\bar{w}$ only when $\bar{w}$ is limited to the subspace WcZ, where $W\equiv\{v\epsilon Z:\ c(w,v)=0\}$. In other words, $\bar{w}$ has to be c-orthogonal to the eigenfunction w. This condition yields a constraint on the unknown vector $\{\bar{w}\}$, i.e.,

$$\{w\}^T[M]\{\bar{w}\}=0 \tag{41}$$

A simple way to solve a linear equation, Eq. (40), with a linear constraint, Eq. (41), is to apply the theorem of Lagrange multipliers to Eq. (40):

$$[A]\{\bar{w}\}+[M]\{w\}\mu=\{F\} \tag{42}$$

where μ is a scalar representing the Lagrange multiplier. The combination of Eq. (41) and (42) yields $n+1$ simultaneous equations for $\{\bar{w}\}$ and μ:

$$\begin{bmatrix}[A] & [M]\{w\}\\ \{w\}^T[M] & 0\end{bmatrix}\begin{Bmatrix}\{\bar{w}\}\\ \mu\end{Bmatrix}=\begin{Bmatrix}\{F\}\\ 0\end{Bmatrix} \tag{43}$$

It is noted that the leading coefficient matrix on the left side of the above equation is symmetric and nonsingular.[26] A variety of numerical methods may be employed to solve Eq. (43). However, in this study, the Gaussian Elimination is applied to solve $\{\bar{w}\}$.

The coefficient α remains to be determined to complete the computation of the eigenvector design derivative:

$$\{w_b\}=\alpha\{w\}+\{\bar{w}\}$$

The coefficient α can be obtained without difficulty by using Eq. (35)

$$\alpha=-\frac{\bar{w}(x_0)}{\gamma}$$

where the notation γ denotes the maximum amplitude of the vibrating beam. Note that in this study, the position where the maximum amplitude occurs is assumed to be independent of the design variable. This is true when the fundamental eigenmode of a beam is symmetric with respect to the midspan point.

Finally, the design derivative of eigenvalue λ_b can be solved by using Eq. (23) or in a matrix form:

$$\lambda_b=\frac{1}{\{w\}^T[M]\{w\}}\left[\frac{3E}{4Q}\{w\}^T[G]\{w\}\cdot\{w\}^T[G]\{w_b\}-\frac{3EQ_b}{2Q^2}\{w\}^T[G]\{w\}\cdot\{w\}^T[G]\{w\}+EI,_b\{w\}^T[K]\{w\}-\lambda m,_b\{w\}^T[M]\{w\}\right] \tag{44}$$

where the notation Q_b is the design derivative of Q, and where $I,_b$ and $m,_b$ are the design derivatives of moment of inertia and mass density, respectively.

V. Numerical Examples

Based upon the numerical algorithm described previously, the numerical examples and results, associated with the nonlinear eigenvalue and eigenvector analysis, as well as the design sensitivity analysis, are presented and discussed in this section.

The finite element model of a uniform beam to be studied is shown in Fig. 2. The beam is discretized into 12 elements. The length L, thickness h, and width b of the beam model are 120 in., 3 in. and 1 in., respectively. The bending stiffness EI, is given as 67.5×10^6 lb-in^2. The mass density is given as 0.85 lb/in. The dimensionless amplitude of vibration γ/g is defined

Table 1 Comparison of ratios (ω/ω_0) for a simply supported beam

	Present work					
γ/g	(Eq. 17)	(Eq. 20)	Woinowsky-Krieger[4]	Mei[10]	Rao[11]	Srinivasan[8]
0	1.0	1.0	1.0	1.0	1.0	1.0
1	1.0897	1.0882	1.0892	1.0844	1.0888	1.0897
2	1.3228	1.3101	1.3178	1.3033	1.3119	1.3220
3	1.6393	1.6010	1.6257	1.5997	1.6022	1.6370
5	2.3848	2.2495	2.3501	2.2881	–	2.3850

as the ratio between the maximum amplitude γ and the radius of gyration of the beam's cross section.

Based on the finite element equations of the nonlinear eigenproblem studied here, Eqs. (17) and (20), the numerical results are listed in Tables 1 and 2 for a simply-supported beam and a clamped-clamped beam, respectively. It is evident the results of the nonlinear eigenvalue analysis are consistent with those reported in the literature.[4,8,10,11]

As for the design sensitivity analysis, the thickness (h) of the cross-section of the uniform beam is considered as the design variable. The accuracy of the design sensitivities of the nonlinear eigenvalue and eigenvector is checked, based on the fundamental definiton of design derivatives which states that they can be approximated by the finite difference. The perturbations of eigenvalues and eigenvectors are, therefore represented by the differences of eigenvalues and eigenvectors evaluated at the perturbed design, b^*, and the current design, b.

The first example presented here deals with the design sensitivity analysis of nonlinear eigenvalues associated with various perturbations of design variable h for a simply supported beam. In this example and the example to follow, the dimensionless amplitude of vibration is fixed at 5, i.e., $\gamma/g = 5$. The nonlinear eigenvalue λ of a uniform beam with the nominal thickness $h = 3''$ is 212.16. For various thicknesses changing from 2″ to 4″, their corresponding nonlinear eigenvalues are listed in Table 3. The comparison of the last two columns in Table 3 demonstrates the validity of the proposed algorithm for the design sensitivity analysis. For the given beam, the values of the fundamental linear eigenvalue and its

Table 2 Comparison of ratios (ω/ω_0) for a clamped-clamped beam

	Present work			
γ/g	(Eq. 17)	(Eq. 20)	Mei[10]	Rao[11]
1	1.0222	1.0211	1.0211	1.0217
2	1.0857	1.0804	1.0816	1.0831
3	1.1833	1.1714	1.1744	1.1756

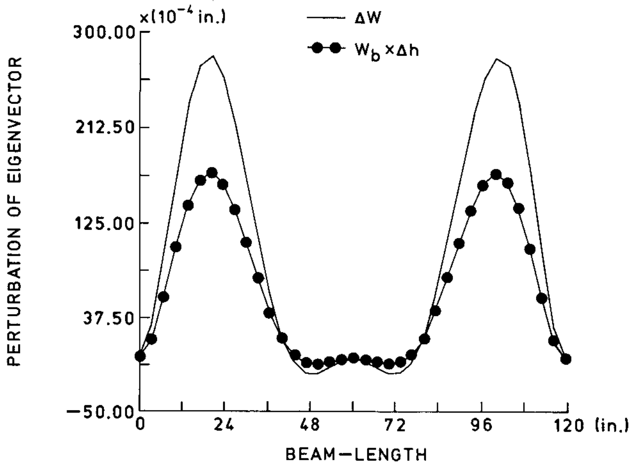

Fig. 3 Design sensitivity analysis of eigenvector associated with perturbation of design variable $\Delta h = 1.0$, for a clamped-clamped beam (Based on Eq. (17).

Table 3 Design sensitivity analysis of eigenvalues associated with various perturbations of design variable for a simply-supported beam

					Actual change	Prediction
h, (in.)	h_c, (in.)	$\Delta h = h_c - h$, (in.)	λ	λ_c	$\Delta\lambda = \lambda_c - \lambda$	$\lambda_b x \Delta h$
3.0	2.0	−1.0	212.1663	191.6754	−20.4910	−24.869
3.0	2.9	−0.1	212.1663	209.8496	−2.3167	−2.4869
3.0	3.1	+0.1	212.1663	214.9797	2.8133	2.4869
3.0	4.0	+1.0	212.1663	241.4236	29.2573	24.869

Table 4 Design sensitivity analysis of eigenvalues associated with various perturbations of design variable for a clamped-clamped beam

					Actual change	Prediction
h, (in.)	h_c, (in.)	$\Delta h = h_c - h$, (in.)	λ	λ_c	$\Delta\lambda = \lambda_c - \lambda$	$\lambda_b x \Delta h$
3.0	2.0	−1.0	401.7889	291.8253	−109.9636	−129.9292
3.0	2.9	−0.1	401.7889	389.0444	−12.7445	−13.0405
3.0	3.1	+0.1	401.7889	415.4528	13.6639	13.5104
3.0	4.0	+1.0	401.7889	553.2861	151.4972	130.9575

Table 5 Design sensitivity analysis of eigenvectors (displacements) associated with various perturbations of design variable for a clamped-clamped beam

					Actual change	Prediction
h, (in.)	h_c, (in.)	$\Delta h = h_c - h$, (in.)	w, [b](in.)	w_c,[b] (in.)	Δw,[b] (in.)	$w_b x \Delta h$,[b] (in.)
3.0	2.0	−1.0	4.10122	4.11880	0.01758	0.01191
3.0	2.9	−0.1	4.10122	4.10251	0.00129	0.00119
3.0	3.1	+0.1	4.10122	4.10005	−0.00117	−0.00119
3.0	4.0	+1.0	4.10122	4.09221	−0.00901	−0.01191
CPU Time[a]					10.89 s[c]	3.21 s[d]

[a]On IBM 4381/CMS, double precision. [b]The perturbation of the displacement of the eigenvector at node 6 (see Fig. 2). [c]Analysis for $h = 2$ in. [d]Design sensitivity analysis for $h = 3$ in.

derivative are both 37.3. Compared to its nonlinear counterpart, the linear eigenvalue is about six times less and its derivative is about 1.5 times larger. Thus, it is interesting to note that the linear eigenvalue is more sensitive to the thickness change of the beam than the nonlinear eigenvalue is.

It is known[13] that the shape of the nonlinear eigenvector remains the same for the vibration of a simply-supported beam, regardless of the values of h. In other words, in theory, $\{w_b\}$ should be a null vector. The numerical study of the example described previously also confirms this. As an example, for the nominal design, $h = 3''$, the maximum perturbation of the eigenvector calculated by $w_b \cdot \Delta h$ is in the order of 10^{-8} for the perturation of thickness $\Delta h = 1''$.

In the next example, the accuracy of the design sensitivity analysis of the nonlinear eigenvalue and eigenvector for a clamped-clamped uniform beam is studied. The nonlinear eigenvalue in this case for the nominal design, $h = 3''$, is 401.79. Similar to the previous example, the comparison between the last two columns in Table 4 shows that the proposed algorithm for the design sensitivity analysis performs very well. The excellent accuracy of nonlinar eigenvector design sensitivity analysis can be observed in Table 5 and Fig. 3.

It is very important to note that, as indicated in the last row of Table 5, the proposed method takes one-third of CPU time to calculate the design derivatives for the nonlinear eigenproblem, compared to the direct difference method.

The detailed numerical study of the proposed algorithm for the calculation of the eigenvalue and eigenvector design derivatives based on Eqs. (20) and (A.2) and (A.3) can be found in Ref. (26). The eigenvector design derivatives for $\gamma/g = 3$ is shown in Fig. 4. Note that even though eigenvalues calculated from nonlinear vibration formulations of Eqs. (19) and (20) are very close to each other, the eigenvector design derivatives are not quite the same. This is a good example to demonstrate that the design derivative information which relates the responses to the system parameters can be used as a tool to verify the mathematical model of a physical phenomenon.

In conclusion, the numerical experiments have shown that the proposed algorithm for calculating design derivatives of nonlinear eigenvalue and eigenvector is valid, and compared with the direct difference method, the proposed algorithm is efficient.

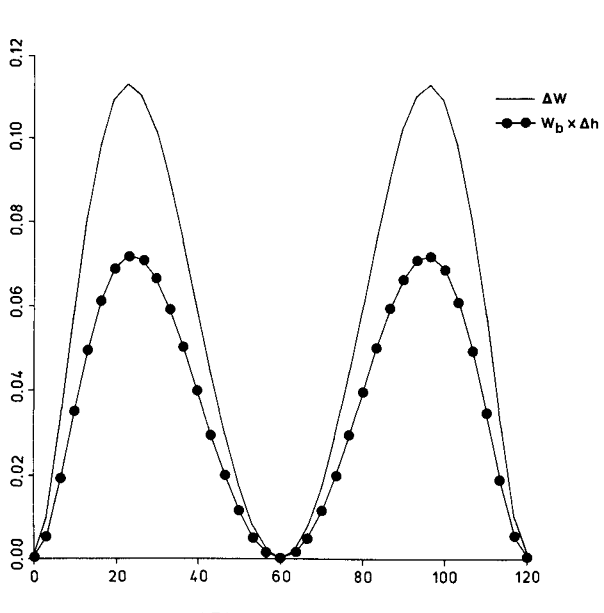

Fig. 4 Design sensitivity analysis of eigenvector associated with perturbation of design variable $\Delta h = 1.0$, for a clamped-clamped beam (Based on Eq. (20).

VI. Conclusion and Remarks

The design derivatives of eigenvalue and eigenfunction in the nonlinear vibration of beams are first presented in an analytic form. The finite element method, the subspace iteration method and the Gaussian elimination method are then employed to find those design derivatives numerically. The validity of the method has been illustrated through the study of examples of a simply supported beam case and a clamped-clamped beam case.

This investigation shows two interesting features of the design sensitivity analysis for nonlinear vibration problems:

1) The design derivatives of the nonlinear eigenvalue and the corresponding eigenfunction are interrelated. The eigenvalue design derivative cannot be calculated without knowing the eigenvector design derivative in advance.

2) Although it starts with a nonlinear equation of vibration, the eigenfunction design derivative can be found by solving a linear and symmetric equation.

Similar to the linear eigenproblem, the $n \times n$ coefficient matrix of the equation for solving the nonlinear eigenvector design derivative is singular. Although the algorithm presented in the report is easily programmed, the coefficient matrix is extended to a $(n+1) \times (n+1)$ symmetric matrix. To store this new matrix, an extra $(2n+2)$ dimension is required. A storage-efficiency method proposed by Nelson[17] for solving the linear eigenvector derivative may be extended to the nonlinear case.

This research represents an initial attempt to find a unified approach for determining the design derivatives of nonlinear eigenvalue and eigenvector. The presented results are limited to two simple formulation of nonlinear vibration. As indicated in Sec. I of the literature survey, the formulation of nonlinear vibration has been developed recently to include the axial deformation and axial inertia. In addition, the various nonlinear vibration problems, associated with different boundary conditions, complex geometries and composite laminates, have been solved. In order to analyze the nonlinear eigenvalue and eigenvector design derivatives of those recently developed formulations, an additional effort is needed to enhance the current study.

Appendix: Design Sensitivity Analysis of Nonlinear Beam Vibrations Based on Mei's Formulation

The variational formula established in Eq. (20) for the nonlinear eigenproblem of a beam can be rewritten here to find an eigenvector $w \epsilon Z$ and an eigenvalue $\lambda \epsilon R^2$ such that

$$\sum_{i=1}^{NE} \left[a_i(w,z) + \frac{EA_i}{4} b_i(w,w) \cdot b_i(w,z) - \lambda c_i(w,z) \right] = 0 \tag{A1}$$

where Z is the kinematically admissible function space. The notations of $a_i(w,z)$, $b_i(w,z)$ and $c_i(w,z)$ have been defined in Eq. (18) with the integration limits changed from $(0,L)$ to $(0,\ell_i)$. Let b be defined as the design parameter that relates to the cross-sectional area. The design derivative of the nonlinear eigenvalue yields the following equation:

$$\lambda_b = 1 \Big/ \sum_{i=1}^{NE} c_i(w,w) \cdot \Bigg\{ \sum_{i=1}^{NE} \Bigg[a_{ib}(w,w) + \frac{EA_{ib}}{4} b_i(w,w) \cdot b_i(w,w) - \lambda c_{ib}(w,w) + \frac{EA_i}{2} \cdot b_i(w_b,w) \cdot b_i(w,w) \Bigg] \Bigg\} \tag{A2}$$

where the second subscript denotes the derivative with respect to the design variable b, i.e., $A_{ib} = \mathrm{d}A_i/\mathrm{d}b$, etc.

Based on the same derivation as discussed in Sec. III, one may find a $\bar{w}\epsilon W$ such that

$$\sum_{i=1}^{NE}\left[a_i(\bar{w},v)+\frac{EA_i}{2}b_i(\bar{w},w)\cdot b_i(w,v)+\frac{EA_i}{4}b_i(w,w)\cdot b_i(\bar{w},v)-\lambda\cdot c_i(\bar{w},v)\right]$$

$$=-\sum_{i=1}^{NE}\left[a_{ib}(w,v)-\frac{EA_{ib}}{4}b_i(w,w)\cdot b_i(w,v)-\lambda\cdot c_{ib}(w,v)\right],\quad \text{for all}\quad v\epsilon W \qquad (A3)$$

where the set W is defined in Sec. III. One can show that the bilinear form on the left side of Eq. (A.3) is positive definite on W. The detailed proof can be found in Ref. (26).

Next, according to Eq. (25), the design derivative of the eigenvector, w_b, can be uniquely obtained as,

$$w_b=\alpha w+\bar{w},\quad \alpha\epsilon R' \quad \text{and} \quad \bar{w}\epsilon W$$

where α is determined by Eq. (35).

Acknowledgments

The authors are grateful to Professors C. Mei (Old Dominion University) and A.H. Neyfeh (Virginia Polytechnic Institute and State University) for their valuable comments.

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